Third-order dissipative hydrodynamics from the entropy principle

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We review the entropy based derivation of third-order hydrodynamic equations and compare their solutions in one-dimensional boost-invariant geometry with calculations by the partonic cascade BAMPS. We demonstrate that Grad's approximation, which underlies the derivation of both Israel-Stewart and third-order equations, describes the transverse spectra from BAMPS with high accuracy. At the same time solutions of third-order equations are much closer to BAMPS results than solutions of Israel-Stewart equations. Introducing a resummation scheme for all higher-oder corrections to one-dimensional hydrodynamic equation we demonstrate the importance of higher-order terms if the Knudsen number is large.

A causal theory of relativistic dissipative hydrodynamics was first formulated by Israel and Stewart [1] and has been successfully applied to study a wide range of ultra-relativistic heavy-ion collision phenomena [2–4]. However, recently presented detailed comparisons of solutions of Israel-Stewart equations and kinetic transport calculations have demonstrated that deviations between them increase with increasing strength of dissipation. Israel-Stewart equations can be derived from the divergence of the off-equilibrium entropy current, which is expanded up to second-order in dissipative fluxes. It is thus of interest to investigate whether a better agreement between hydrodynamic and kinetic transport calculations can be achieved if the entropy current is expanded one order higher than in Israel-Stewart theory.

The underlying equation of the kinetic transport theory is the Boltzmann equation

$$p^\mu \partial_\mu f(x,p) = C[f(x,p)] \tag{1}$$

which describes the space-time evolution of the phase-space particle distribution $f(x,p) = \frac{dN}{d^3p d^3x}$ due to drift and diffusion on the left hand side and the collision processes on the right hand side of the equation. A connection between the microscopic kinetic transport theory and the macroscopic theory of hydrodynamics can be established using the Grad's method, in which the off-equilibrium distribution is approximated by

$$f_{off-eq} = f_0 \left(1 + \epsilon + \epsilon_\mu p^\mu + \epsilon_{\mu\nu} p^\mu p^\nu\right) \tag{2}$$

where f_0 denotes the isotropic (equilibrium) distribution and the fields ϵ, ϵ_μ and $\epsilon_{\mu\nu}$ are related to the dissipative fluxes Π, q^μ and $\pi^{\mu\nu}$. The exact form of ϵ, ϵ_μ and $\epsilon_{\mu\nu}$ can be obtained from the definitions of dissipative currents and the matching conditions

$$u_\mu u_\nu (T^{\mu\nu} - T_{eq}^{\mu\nu}) = 0, \quad u_\nu(N^\nu - N_{eq}^\nu) = 0 \tag{3}$$

where $T_{eq}^{\mu\nu}$ and N_{eq}^ν are the energy-momentum tensor and the particle current in equilibrium.

In the following we consider a massless gas of Boltzmann particles (gluons) undergoing a boost-invariant one dimensional expansion [5]. For the considered system the equation of state is $e = 3p$. Bulk pressure and heat flux vanish identically and the local rest frame off-equilibrium distribution becomes [9, 10]

$$f(x,p) = d_g e^{-E/T} \left(1 + \frac{3}{8eT^2} \pi_{\mu\nu} p^\mu p^\nu\right) \tag{4}$$

with the energy density e and the degeneracy factor for gluons, $d_g = 16$. The matching conditions (3) allow to define the temperature T for an off-equilibrated system by matching its energy and particle densities e and n to a fictitious equilibrium state:

$$e = e_{eq}, \quad n = n_{eq} \tag{5}$$

For the Boltzmann gas considered here the temperature T in Eq.(4) is then

$$T = \frac{e}{3n}. \tag{6}$$
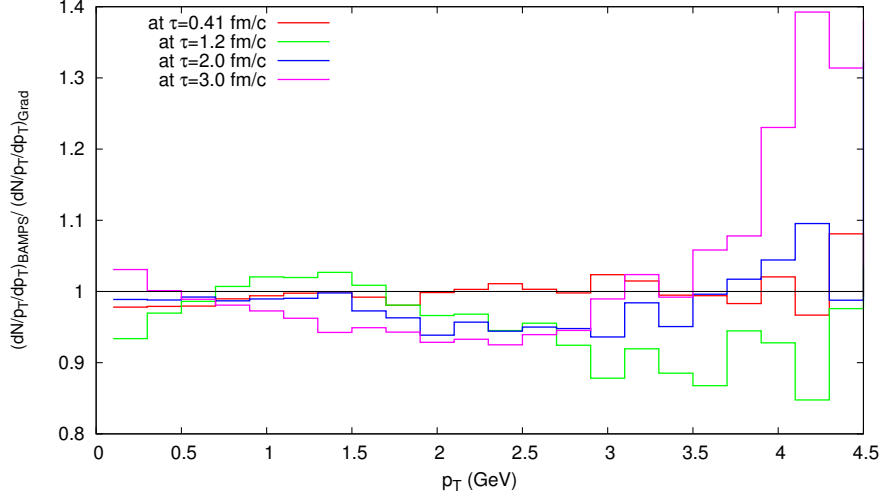
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FIG. 1: Ratio of transverse particle distribution from BAMPS with $\eta/s = 0.4$ to the one calculated by Eq.(4) at different times with e, T, $\pi^{\mu\nu}$ extracted from BAMPS.

Grad's approximation, Eqs.(2) resp. (4), is essential for derivations of hydrodynamic equations from the Boltzmann equation [15, 20] resp. from the entropy principle [6, 10]. It is thus important to quantify how accurate the approximation in Eq.(4) can reproduce the off-equilibrium distribution obtained from the numerical solution of the Boltzmann Equation in the partonic cascade BAMPS [17]. BAMPS has recently been applied to investigate a wide range of phenomena such like the buildup of the elliptic flow [18], the energy loss of high energy gluons [19], the extraction of the second-order viscosity coefficient [9], and the formation and propagation of shock waves [14] in ultra-relativistic heavy-ion collisions.

For the results presented in this work we use BAMPS calculations with elastic isotropic cross section adjusted in such a way that the shear viscosity to entropy density ratio is constant throughout the evolution, as introduced in [8, 10]. The initial condition is a Boltzmann distribution with $T_0 = 500 MeV$ at the initial time $\tau_0 = 0.4 fm/c$. All results are extracted from the central rapidity bin $\eta \in [-0.1 : 0.1]$.

To quantify the deviations of the expression in Eq.(4) from the actual distribution in BAMPS we take the ratio δ_{Grad}^{BAMPS} of the transverse distributions calculated as follows:

$$\delta_{Grad}^{BAMPS} = \frac{(dN/p_T/dp_T)_{BAMPS}}{(dN/p_T/dp_T)_{Grad}} = \frac{\left\langle p_0 f_{BAMPS} \right\rangle_{y,\varphi}}{\left\langle p_0 d_g \lambda e^{\frac{-p_T \cosh y}{T}} \left(1 + \frac{3}{8T^2} \frac{\pi}{e} p_T^2 \left(\frac{1}{2} - \sinh^2 y\right)\right) \right\rangle_{y,\varphi}}. \tag{7}$$

In the latter expression π denotes the shear pressure, which is the dissipative correction to the longitudinal pressure:

$$\pi = -\pi^{33} = T_{eq}^{33} - T^{33} \tag{8}$$

For the analytic calculation using Eq.(4) the values of e and T as well as the components of $\pi_{\mu\nu} = T_{\mu\nu} - T_{\mu\nu}^{eq}$ are extracted from BAMPS. The deviation δ_{Grad}^{BAMPS} is shown in Fig.1 at different times as function of p_T. The deviations of Grad's approximation from actual distribution in BAMPS do not exceed 10% for the chosen value of η/s below $p_T \sim 3 GeV$. The good agreement of BAMPS distribution with Grad's approximation observed in Fig.1 might indicate that the analytic expression in Eq.(4) can be applied for derivations of hydrodynamic equations.

Derivation of the evolution equation for the shear stress tensor has been reported by us recently in [10]. The starting point for the derivation is the entropy current which in kinetic theory can be written as

$$s^\mu = -\int \frac{d^3p}{E} p^\mu f (\ln f - 1). \tag{9}$$

Writing the off-equilibrium distribution as $f = f_0 (1 + \phi)$ (comp. Eqs. (2) and (4)) we expand the logarithm up to third order in ϕ and obtain by a direct calculation

$$s^\mu \approx s_0 u^\mu - \frac{\beta_2}{2T} \pi_{\alpha\beta} \pi^{\alpha\beta} u^\mu - \frac{8}{9} \frac{\beta_2^2}{T} \pi_{\alpha\beta} \pi_\sigma^\alpha \pi^{\beta\sigma} u^\mu \tag{10}$$

with $s_0 = 4n - n \ln \lambda$ and $\beta_2 = \frac{9}{4e}$. Note that truncating the expansion at order ϕ^2 we obtain the entropy current from the Israel-Stewart theory [1, 6].

The original Israel-Stewart equations explicitly satisfy the second law of thermodynamics since they are obtained directly from the requirement of non-negativeness of the divergence of the entropy current, $\partial_\mu s^\mu \geq 0$. Using the same argumentation with a third-order entropy current in Eq. (10), i.e. calculating its divergence and imposing a linear relation between the dissipative flux and the corresponding thermodynamic force, we obtain an extended version of Israel-Stewart's equation for $\pi^{\mu\nu}$ in which we keep only terms up to third-order in Knudsen number $Kn \sim \tau_\pi \partial_\mu u^\mu$ resp. dissipative fluxes [10]:

$$\dot{\pi}^{\alpha\beta} = -\frac{\pi^{\alpha\beta}}{\tau_\pi} + \frac{\sigma^{\alpha\beta}}{\beta_2} - \pi^{\alpha\beta}\frac{T}{\beta_2}\partial_\mu\left(\frac{\beta_2}{2T}u^\mu\right) - \frac{8}{9}\frac{T}{\beta_2}\partial_\mu\left(\frac{\beta_2^2}{T}u^\mu\right)\pi_\sigma^{\langle\alpha}\pi^{\sigma\beta\rangle} \tag{11}$$

The latter equation constitutes a novel third-order evolution equation for the shear tensor. The notation $\dot{\pi}$ denotes derivative with respect to the proper time τ. τ_π is the relaxation time and is the same as in Israel-Stewart theory:

$$\tau_\pi = 2\eta\beta_2. \tag{12}$$

Neglecting the last two terms in Eq. (11) the second-order Israel-Stewart equation is recovered. To second order the equation we obtain does not contain all terms found in recent works [3, 12, 13, 15]. It is an interesting task for the future to understand the differences between various formulations of dissipative hydrodynamic equations, as discussed for example in [15]. We have to stress that in our derivations, as presented here and in more detail in [10], we do not use the Boltzmann Equation directly. The terms we obtain are the full set of terms which can be obtained from the entropy principle if Grad's approximation, given by Eq. (4), is used.

For a one-dimensional system with boost-invariance Eq. (4) takes the following form

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} - 3\frac{\pi^2}{e\tau}. \tag{13}$$

In the latter equation π denotes the shear pressure which reduces the longitudinal pressure p_L:

$$p_L = T_{33} = p - \pi. \tag{14}$$

Again, the Israel-Stewart equation is recovered from Eq. (13) if the last term is neglected. Eq. (13) has to be solved together with the evolution equations for the energy and particle densities. The former is obtained from the conservation of the energy-momentum tensor component, $\partial_\nu T^{\nu 0} = 0$. The latter we obtain assuming conservation of the particle number current, $\partial_\mu N^\mu = 0$, i.e. assuming a medium in which net particle number is constant. In the geometry chosen here, the evolution equation for the energy and particle densities read

$$\dot{e} = -\frac{4}{3}\frac{e}{\tau} + \frac{\pi}{\tau} \quad , \quad \dot{n} = -\frac{n}{\tau}. \tag{15}$$

Before discussing the solutions of Eq. (13) resp. of the Israel-Stewart's equations we would like to discuss the effect of higher than 3rd order contributions to Eq. (13). This analysis will be presented here for a one-dimensional system. In order to include all orders of corrections into Eq. (13) we assume they all have the form $x_n\left(\frac{\pi}{e}\right)^n\frac{e}{\tau}$ with $n \geq 3$. Note that the second and third-order terms in Eq. (13) are already of this form. Thus, an *ansatz* for an equation containing all orders of corrections can be written in the following way

$$\begin{aligned}
\dot{\pi} &= -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} + \sum_2^\infty x_n\left(\frac{\pi}{e}\right)^n\frac{e}{\tau} = \\
&= -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} + \frac{\pi^2}{e\tau}\underbrace{\sum_2^\infty x_n\left(\frac{\pi}{e}\right)^{n-2}}_{\chi} = \\
&= -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} + \chi\frac{\pi^2}{e\tau}.
\end{aligned} \tag{16}$$

The coefficient χ is supposed to be an unknown function of time. Since the equation we consider is supposed to include all orders of corrections, it should be applicable in the free-streaming, i.e. $\tau_\pi \to \infty$, limit. In the free-streaming limit the solutions for e, π and n are known. Since the gas is streaming free, longitudinal pressure cannot be built up, i.e.
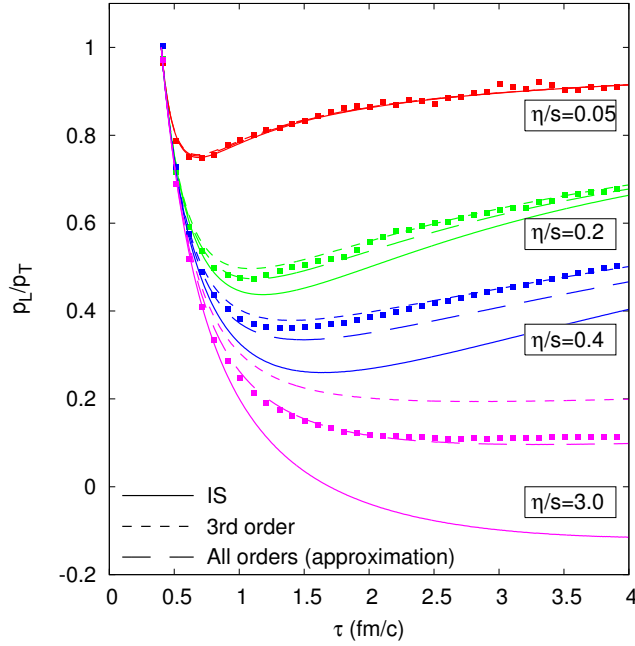
FIG. 2: Time evolution of pressure isotropy. The shear pressure is calculated by Eqs. (15) and (16) using $\chi = 0$ (Israel-Stewart), $\chi = -3$ (extension of Israel-Stewart's equation to third order) and $\chi = -\frac{5}{3}$ (approximate inclusion of all order of corrections).

$p_L = 0$, which means $\pi = p = \frac{e}{3}$. Using this, one finds that the energy density evolves according to Eq. (15) as $e(\tau) = e_0\tau_0/\tau$. Using these solutions in the $\tau_\pi \to \infty$ limit in Eq. (16) one obtains

$$\chi = -\frac{5}{3}. \qquad (17)$$

This value is the result of resummation of higher-order terms in the heuristic *ansatz* Eq. (16). The equation obtained this way includes all orders of corrections, but only approximately. If only third-order terms are included, one obtains $\chi = 3$, which corresponds to Eq. (13). The Israel-Stewart equation is obtained by setting $\chi = 0$.

In the following we present the solutions of hydrodynamic equation of second (Eq. (16) with $\chi = 0$) and third orders (Eq. (16) with $\chi = -3$) and of the approximation of all orders (Eq. (16) with $\chi = -\frac{5}{3}$). The observable we use to quantify the deviations from equilibrium is the pressure isotropy $\frac{p_L}{p_T} = \frac{p-\pi}{p+\pi/2}$. The results are compared to BAMPS calculations using thermal initial conditions with $T_0 = 0.5 GeV$ and $\tau_0 = 0.4 fm/c$. For the comparisons presented here only elastic processes with isotropic cross section are included in BAMPS. The cross section σ_{22} is parametrized to keep the η/s value constant, as has been already done in [8, 10, 14]:

$$\sigma_{22} = \frac{6}{5}\left(\frac{\eta}{s}\right)^{-1}\frac{T}{4n - n\ln\lambda}. \qquad (18)$$

where $4n - n\ln\lambda = s$. The method of derivation of hydrodynamic equations which has been presented here does not allow to obtain an analytic expression for the shear viscosity coefficient. We thus rely on the expression for η obtained in [21] which is strictly speaking valid for the Israel-Stewart, i.e. second-order, theory.

Time evolution of the pressure isotropy is presented in Fig.2. It was demonstrated in [14] that the evolution of the system is governed by the Knudsen number $Kn \equiv \tau_\pi/\tau \approx 6\frac{1}{T\tau}\frac{\eta}{s}$. Since $T \approx T_0\left(\tau_0/\tau\right)^{\frac{1}{3}}$, the Knudsen number depends in the situation considered here on τ_0, τ, T and η/s. Since all results are obtained using the same initial conditions, the evolution depends only on the chosen value of η/s, which is thus the direct measure of the strength of dissipative effects. The deviations between the Israel-Stewart and BAMPS results become considerable already at $\eta/s = 0.4$. At $\eta/s = 3$ the Israel-Stewart's equations ($\chi = 0$ in Eq. (16)) lead to a negative longitudinal pressure which is not physical for the considered setup (This phenomenon has been as well investigated in [7]). The absolute deviations are reduced if one extends Israel-Stewart's equations to third-order, Eq. (13) resp. $\chi = -3$ in Eq. (16). Moreover, the negative longitudinal pressure does not occur in the solutions of the third-order equations. Especially at late times, where the relaxation towards equilibrium sets in, the third-order results on pressure isotropy are in very

good agreement with BAMPS solutions. At early times the third-oder and BAMPS results still deviate since there the expansion scalar $\partial_\mu u^\mu = 1/\tau$, is still large and thus all orders of corrections have to be taken into account. Indeed, at early times solution of the approximate all-order equation, $\chi = -\frac{5}{3}$ in Eq. (16), is in very good agreement with BAMPS results. This is due to the fact that the value $\chi = -\frac{5}{3}$ has been obtained from the $\tau_\pi \to \infty$ or alternatively $Kn \to \infty$ limit, to which the system is close at early times.

In this study we have presented an extension of the Israel and Stewart's entropy based approach to third order in dissipative fluxes. For this study we have considered a one-dimensional boost-invariant gas of massless Boltzmann particles (gluons). Results of hydrodynamic calculations have been compared to solutions of the Boltzmann Equation from the partonic cascade BAMPS. Although the Grad's approximation, upon which the derivation of Israel-Stewart's equation is based, has been shown to describe transverse spectra from BAMPS with remarkable accuracy, the solutions of Israel-Stewart equations demonstrate large deviations from BAMPS. Inclusion of third-order terms into the evolution equation for shear tensor reduces the deviations between hydrodynamic and BAMPS results considerably. Up to $\eta/s = 0.4$ the third-order solutions are in very good agreement with BAMPS. In order to estimate the effect of all orders of correction, we have introduced a resummation scheme which allows to represent the infinite series of higher-order terms by one single term. The solution of this equation is in very good agreement with BAMPS results even at $\eta/s = 3$, when the Knudsen number is very large, which underlines the importance of higher-order corrections at early times of evolution. It is still an important task for future studies to understand the differences between the equations obtained here from the entropy principle and the second-order equations presented in recent publications by different authors [3, 13, 15, 20] . It is as well important to investigate whether a possible extension of Grad's approximation can further improve the agreement between hydrodynamic and kinetic transport calculations.

Acknowledgements

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